A NEW DETERMINATION OF THE PRIMITIVE CONTINUOUS GROUPS IN TWO VARIABLES*

BY

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The primitive continuous groups of point-transformations in two variables can, by a proper choice of the variables, be transformed into *projective groups* of the plane, a result LIE obtains after determining the canonical forms of the primitive groups.† This fact can, however, be established from the general properties of such groups, and its use leads to a new determination of these primitive groups, to show which is the object of this paper.

A primitive group will be defined as a group which does not leave invariant a differential equation of the first order.‡ Such a group is at least three-parametric, as a two-parametric group possesses a differential invariant of the first order, J say, and therefore an invariant differential equation of the first order, $f(J) =$ constant.§

§ 1.

It will first be necessary to show that *any group in two variables and of more than two parameters leaves invariant at least one differential equation, which is integral and algebraic in the derivatives* dy/dx, d^2y/dx^2, *etc., involved.*

Let $Xf \equiv \xi\, \partial f/\partial x + \eta\, \partial f/\partial y$ indicate any one of the infinitesimal transformations of the group. If we "extend" these n times,|| and write $y_1, y_2, \cdots, y_n$ for $dy/dx, d^2y/dx^2, \cdots, d^ny/dx^n$, and $p, q, p_1, p_2, \cdots, p_n$ for $\partial f/\partial x, \partial f/\partial y, \partial f/\partial y_1, \partial f/\partial y_2, \cdots, \partial f/\partial y_n$, respectively, we easily find these extended transformations to be of the form

$$\xi p + \eta q + [\eta_x + (\eta_y - \xi_x) y_1 - \xi_y y_1^2] p_1 + (P_2 y_2 + Q_2) p_2 + \cdots + (P_n y_n + Q_n) p_n,$$

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† S. LIE: *Vorlesungen über continuierliche Gruppen*, herausgegeben von Dr. SCHEFFERS, p. 359. This book will hereafter be designated *Continuierliche Gruppen.*

‡ *Invariant differential equation* defined in *Continuierliche Gruppen*, pp. 213–214, and in *Differentialgleichungen mit bekannten infinitesimalen Transformationen*, by S. LIE, p. 277.

§ *Continuierliche Gruppen*, p. 228.

|| *Continuierliche Gruppen*,, pp. 213–214.

where

$$P_m = -(m+1)\xi_y y_1 - m\xi_x + \eta_y,$$

$$Q_2 = -\xi_{yy}y_1^3 + (-2\xi_{xy} + \eta_{yy})y_1^2 + (-\xi_{xx} + 2\eta_{xy})y_1 + \eta_{xx},$$

$$Q_3 = -3\xi_y y_2^2 + y_2 R_3(x, y, y_1) + S_3(x, y, y_1),$$

$$Q_m = -\tfrac{1}{2}m(m-1)\xi_y y_2 y_{m-1} + y_{m-1}R_m(x, y, y_1)$$
$$+ S_m(x, y, y_1, \cdots, y_{m-2}) \quad (m > 3),$$

the quantities R and S being integral algebraic functions of $y_1, y_2, \cdots, y_{m-2}$.

We shall indicate the n times extended infinitesimal transformation Xf by

$$X^n f \equiv \xi p + \eta q + \sum_{i=1}^{n} \eta^{(i)} p_i.$$

Lie has proved that an m-parametric group possesses one, and only one, differential invariant, J, of order less than m.*

First, let J be of order $m-1$. Then must the m equations

$$(1) \qquad X^{m-2}f \equiv \xi\frac{\partial f}{\partial x} + \eta\frac{\partial f}{\partial y} + \sum_{i=1}^{m-2}\eta^{(i)}\frac{\partial f}{\partial y_i} = 0$$

be independent of each other, $\xi\,\partial f/\partial x + \eta\,\partial f/\partial y$ being a type of the m independent infinitesimal transformations of the group. For, were they not independent, we should have a complete system of fewer than m equations in the m variables $x, y, y_1, \cdots, y_{m-2}$, which would have as solution a differential invariant of degree $m-2$ at most, contrary to hypothesis.

The determinant D of the m^2 coefficients $\xi, \eta, \eta^{(i)}$ of the equations (1) will then not vanish identically. If it contains any one of the variables $y_1, y_2, \cdots, y_{m-2}$, then will $D = 0$ be an invariant differential equation,† and by examining the form of the quantities $\eta^{(i)}$ given above, we find that D must be an integral algebraic function of $y_1, y_2, \cdots, y_{m-2}$.

The determinant D may be a function of x, y only, say $\phi(x, y)$, which is easily seen to be the determinant of the m^2 quantities $\xi, \eta, \eta_x, \eta_{xx}, \cdots, \partial^{m-2}\eta/\partial x^{m-2}$.‡ Now, by choosing y for the independent variable instead of x, the transformed determinant D of (1) will be seen to contain $\pm\phi(dx/dy)^{(m+1)(m-2)/2}$ as the highest power of dx/dy. In this case then, *if* $m > 2$, $D = 0$ will be an invariant differential equation, integral and algebraic in the derivatives involved.

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* *Continuierliche Gruppen*, page 230. In the text J is given by J_{m-1}.

† S. Lie, *Theorie der Transformationsgruppen*, bearbeitet unter Mitwirkung von Dr. F. Engel; I. Abschnitt, Kapitel 14.

‡ These quantities are the terms of $X^{m-2}f$ free from the derivatives $y_1, y^2, \cdots, y_{m-2}$.

Next, let J be of less than the $(m-1)$-th order. Every differential equation $J=$ constant is invariant, and therefore the equation $dJ/dx=0$ is invariant and is of the nth order, say, where $n \leqq m-1$.

The various equations:

$$(2)\qquad X^n\left(\frac{dJ}{dx}\right) \equiv \xi\frac{\partial}{\partial x}\left(\frac{dJ}{dx}\right) + \eta\frac{\partial}{\partial y}\left(\frac{dJ}{dx}\right) + \sum_{i=1}^{n}\eta^{(i)}\frac{\partial}{\partial y_i}\left(\frac{dJ}{dx}\right) = 0$$

cannot be identities, since dJ/dx is not a differential invariant, there being only one such of order less than m, which is here J. Since this is of the $(n-1)$-th order, only n of the equations

$$\xi\frac{\partial f}{\partial x} + \eta\frac{\partial f}{\partial y} + \sum_{i=1}^{n-1}\eta^{(i)}\frac{\partial f}{\partial y_i} = 0$$

can be independent, and it is therefore possible to eliminate the quantities

$$\frac{\partial}{\partial x}\left(\frac{dJ}{dx}\right),\quad \frac{\partial}{\partial y}\left(\frac{dJ}{dy}\right),\quad \frac{\partial}{\partial y_i}\left(\frac{dJ}{dx}\right) \qquad (i=1, 2, \cdots, n-1),$$

in various ways from (2) and thus to obtain equations of the form

$$M\frac{\partial}{\partial y_n}\left(\frac{dJ}{dx}\right) = 0,$$

where the quantities M are determinants of the coefficients $\xi, \eta, \eta', \cdots, \eta^{(n)}$ and are not all zero identically, but must evidently all vanish on account of $dJ/dx=0$. Since the determinants M are integral and algebraic in $y_1, y_2, \cdots, y_n$, this must be the case with $dJ/dx=0$, after a possible reduction.

In all cases then, we have an invariant differential equation, integral and algebraic in the derivatives involved, as stated above.

§ 2.

Let now

$$\phi \equiv Ay_n^a + By_n^{a-1} + \cdots = 0$$

be such an equation of least order n, left invariant by a *primitive group*, which, according to definition, does not leave invariant a differential equation of the first order. It is supposed that ϕ does not contain any factor which is rational in $y_1, \cdots, y_n$. *We can show that* $n=2$.

The conditions for invariance are that $X^n\phi=0$ on account of $\phi=0$ for every infinitesimal transformation Xf of the group. Hence, as

$$X^n f \equiv (P_n y_n + Q_n)\frac{\partial f}{\partial y_n} + X^{n-1} f \qquad (n>1),$$

we have

$$AX^n\phi \equiv \phi(aAP_n + X^{n-1}A).$$

By taking the $(a-1)$-th partial derivative with respect to y_n of both members of these identities we get, after reduction,

$$(3) \qquad AX^n(aAy_n + B) \equiv (aAy_n + B)(AP_n + X^{n-1}A).$$

The equation

$$aAy_n + B = 0$$

is therefore invariant. We may suppose that A and B have no factors in common containing derivatives.

Then follows from (3):

$$(4) \qquad BX^{n-1}A \equiv A(aAQ_n + X^{n-1}B - BP_n).$$

Thus $A = 0$ is an invariant differential equation of less than the n-th order, contrary to hypothesis, unless A is a function of x, y only. Let us therefore put $B = aAC$, C being integral in $y_1, y_2, \cdots, y_{n-1}$. The invariant equation is therefore reducible to

$$y_n + C = 0,$$

and the equations (4) reduce to

$$(5) \qquad -P_nC + Q_n + X^{n-1}C \equiv 0.$$

Suppose $n > 3$. Putting $C = \alpha y_{n-1}^t + \beta y_{n-1}^{t-1} + \cdots$, we find that, if $t > 1$,

$$(tP_{n-1} - P_n)\alpha + X^{n-2}\alpha = 0,$$

so that $\alpha = 0$ would be an invariant differential equation, unless $\alpha \equiv 0$. Indeed, α could not be a function of x, y merely, as not every

$$tP_{n-1} - P_n \equiv \xi_y y_1 + \xi_x - (t-1)(n\xi_y y_1 + n\xi_x - \eta_y)$$

could be free from y_1 for a primitive group, where not every ξ is a function of x alone.*

Thus,

$$C \equiv \alpha y_{n-1} + \beta.$$

Substituting in (5) we find the conditions

$$(6) \qquad (P_{n-1} - P_n)\alpha - \frac{n(n+1)}{2}\xi_y y_2 + R_n(x, y, y_1) + X^{n-2}\alpha \equiv 0.$$

By assuming α, which is integral in $y_1, y_2, \cdots, y_{n-2}$, to be of the form

$$\gamma y_k^s + \delta y_k^{s-1} + \cdots,$$

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*If every ξ were a function of x only, the group would leave invariant a differential equation of the first order, namely, $dx/dy = 0$.

y_k being the derivative of highest order contained in a, and substituting in (6), we find that $k = 2$ and $s = 1$, besides the conditions

$$(7) \qquad \gamma(-P_n + P_{n-1} + P_2) - \frac{n(n+1)}{2}\xi_y + X^1\gamma = 0 .$$

Let

$$\gamma = \epsilon y_1^r + \zeta y_1^{r-1} + \cdots .$$

Substituting in (7) and selecting the coefficients of y_1^{r+1} we get the equations

$$-(2+r)\xi_y\epsilon = 0 ,$$

which are impossibilities for every ξ_y that is not zero. It is to be noticed that ϵ could not be zero and therefore also γ, as then (7) would be impossible.

Thus $n > 3$ is not the least order of an invariant differential equation of the required form for a primitive group.

The case $n = 3$, when treated in the same manner, will give similar results. The coefficients of γ and ξ_y in the equations corresponding to (7) will here be slightly different from those in (7).

Thus $n = 2$. Equations (5) are here

$$-P_2C + Q_2 + X^1C \equiv 0 ,$$

from which we find

$$C \equiv ay_1^3 + \beta y_1^2 + \gamma y_1 + \delta ,$$

a, β, γ, δ being functions of x, y only. The invariant differential equation is therefore

$$(8) \qquad y_2 + ay_1^3 + \beta y_1^2 + \gamma y_1 + \delta = 0 ,$$

the reduction of which is our next object.

§ 3.

A three-parametric group, whose infinitesimal transformations are indicated by $Xf \equiv \xi p + \eta q$, leaves invariant a differential equation of the first order. For, if the determinant of the nine quantities ξ, η, η' is not zero identically, we have such an invariant equation as explained in the beginning of § 1, which will be of the first order in this case; and if the determinant vanishes identically, we have at least one differential invariant of the first order, J say, and therefore an invariant equation $f(J) = 0$.

A primitive group must therefore be at least four-parametric. Now, Dr. F. Engel has shown* that a group with more than three parameters contains at least one pair of independent infinitesimal transformations which are permutable.

* *Zur Theorie der Zusammensetzung der endlichen continuirlichen Transformationsgruppen*, Leipziger Berichte, 1886, pp. 83–94.

Such a pair, forming a two-parametric subgroup, can, by a proper choice of the variables, be given by

$$q\,,\, xq\,; \quad \text{or} \quad p\,,\, q\,.^*$$

In the first case it appears from a theorem by LIE† that the equation (8) can, by a transformation of the form

$$\bar{x} = x, \quad \bar{y} = y - \phi(x),$$

be changed into the following:

$$\bar{y}_2 = 0\,.$$

This transformation will change q, xq into $\bar{q}$, $\bar{x}\bar{q}$, where $\bar{q}$ stands for $\partial f/\partial \bar{y}$.

In the second case, by imposing upon the equation (8) the conditions that it should be left invariant by the infinitesimal transformations p, q, we find that the coefficients α, β, γ, δ are constants. Then, if $\alpha \neq 0$, by choosing for new variables

(9) $$x = y - sx, \quad \bar{y} = x,$$

where $$s^3\alpha + s^2\beta + s\gamma + \delta = 0,$$

we get a differential equation of the form

(10) $$\bar{y}_2 + \alpha'\bar{y}_1^2 + \beta'\bar{y}_1 + \gamma' = 0$$

in place of (8), α', β', γ' being constants.

The complete integral of this equation is readily found, and can in all cases be thrown into the form

$$\phi(\bar{x}, \bar{y}) + k_1\psi(\bar{x}, \bar{y}) + k_2 = 0,$$

k_1 and k_2 being the constants of integration. Then by changing to new variables x, y according to the relations

(11) $$y = \phi(\bar{x}, \bar{y}), \quad x = \psi(\bar{x}, \bar{y}),$$

the equation (10) will reduce to

$$y_2 = 0\,.$$

Applying the combined transformations (9) and (11) to the subgroup p, q, we find this changed into one or other of the following subgroups, corresponding to the different forms of the functions ϕ and ψ of (11):

$$p + axq\,,\, q\,; \quad xp\,,\, yq\,; \quad p\,,\, yq\,; \quad xp\,,\, q\,;$$

a being a constant.

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* *Differentialgleichungen mit bekannten infinitesimalen Transformationen*, p. 425.

† *Differentialgleichungen mit* etc., p. 429.

Hence, *by a proper transformation of the variables* x, y, *the primitive groups of the plane can be transformed into groups leaving invariant the differential equation*

$$\frac{d^2y}{dx^2} = 0,$$

and possessing one or other of the following subgroups:

$$q, xq; \quad q, p + axq; \quad xp, yq; \quad p, yq; \quad xp, q.$$

On account of the invariance of the equation $d^2y/dx^2 = 0$, *these transformed groups are projective.**

§ 4.

The problem of finding the primitive groups of the plane is thus reduced to one of finding all the different subgroups of the general projective group

$$p, \; q, \; yp, \; xq, \; xp, \; yq, \; x^2p + xyq, \; xyp + y^2q$$

which do not leave invariant a differential equation of the first order, and which contain the two-parametric subgroups given above.

Before we proceed it might be well to state a few well known propositions about the infinitesimal transformations of a group.

A set of r independent infinitesimal transformations generate an r-parameter group when, and only when, the commutator (Klammerausdruck) of any pair of these, as Xf and Yf:

$$(Xf, Yf) \equiv X(Yf) - Y(Xf),$$

is expressible as a sum of the given infinitesimal transformations multiplied by constants.

If Xf and Yf are any two infinitesimal transformations of a group, then is $aXf + bYf$ also an infinitesimal transformation of the group, a and b being constants.

A primitive group, which is projective, cannot leave invariant a given point in the plane, for otherwise the differential equation of the first order defining the ∞^1 straight lines through that point would be invariant.

Now, let the primitive group required contain the subgroup q, xq. The transformation

$$x = \frac{x'}{y'}, \quad y = \frac{1}{y'}$$

does not alter the form of the differential equation $d^2y/dx^2 = 0$ and it changes the

* *Continuierliche Gruppen*, p. 35.

subgroup q, xq into the subgroup $x^2p + xyq, xyp + y^2q$, after we drop the accents.

One of the additional infinitesimal transformations of the group must be of the form

$$ap + bq + (cx + ey)p + (gx + hy)q \equiv Xf,$$

where a and b are not both zero, since it is necessary to have at least one infinitesimal transformation which does not leave invariant the point $(0, 0)$.

A linear transformation of the variables can now be chosen such that $ap + bq$ is changed into p and the subgroup $x^2p + xyq, xyp + y^2q$ is left unaltered, besides the differential equation $d^2y/dx^2 = 0$.

If now

$$Xf \equiv p + (cx + ey)p + (gx + hy)q,$$

we find by forming the following commutators

$$(Xf, x^2p + xyq) \equiv 2xp + yq + \alpha(x^2p + xyq) + \beta(xyp + y^2q) \equiv Yf,$$

$$(Xf, xyp + y^2q) \equiv yp + \alpha'(x^2p + xyq) + \beta'(xyp + y^2q),$$

$$\big((Xf, Yf) + Xf, Yf\big) \equiv 6p + 3\alpha(2xp + yq) + (4\beta + 2e)yp + \alpha''(x^2p + xyq) + \beta''(xyp + y^2q),$$

that the group required must contain the infinitesimal transformations

$$p, \; yp, \; 2xp + yq, \; x^2p + xyq, \; xyp + y^2q,$$

which will be found to generate a group. This group does not leave invariant a differential equation of the first order and is therefore primitive.

If the required group contains more infinitesimal transformations, we can assume these to be of the form $(a + bx + cy)q$. By building commutators as above we find that two cases present themselves, viz:

one additional infinitesimal transformation only, namely, yq;

three additional infinitesimal transformations, q, xq, yq.

We have now three types of primitive groups, which, by the transformation

$$x = \frac{x'}{y'}, \quad y = \frac{1}{y'},$$

become the following (dropping the accents):

$$\begin{aligned} &p, \; q, \; yp, \; xq, \; xp - yq; \\ (12) \qquad &p, \; q, \; yp, \; xq, \; xp, \; yq; \\ &p, \; q, \; yp, \; xq, \; xp, \; yq, \; x^2p + xyq, \; xyp + y^2q. \end{aligned}$$

The two subgroups p, yq; xp, q are transformed one into the other by exchanging the variables. We need therefore consider only one of these, as p, yq.

The transformation $x = x'/y'$, $y = 1/y'$, used above, changes this subgroup into yp, $xp + yq$, after dropping the accents. As in the case considered above, we must have an infinitesimal transformation of the form

$$ap + bq + (cx + ey)q + g(x^2p + xyq) + h(xyp + y^2q) \equiv Xf,$$

where a and b are not both zero. Then from

$$\big(xp + yq, (xp + yq, Xf) - Xf\big) \equiv 2(ap + bq),$$
$$(yp, ap + bq) \equiv -bp,$$

we see that the required group must contain the infinitesimal transformation p, whether b be zero or not. We have now the subgroup p, yp which, by exchanging the variables, becomes the subgroup q, xq. This has been dealt with above.

Next, let the required group contain the subgroup xp, yq. We must have an infinitesimal transformation of the form

$$ap + bq + cyp + exq + g(x^2p + xyq) + h(xyp + y^2q) \equiv Xf,$$

a and b not being both zero.

From the commutators

$$\big(xp + yq, (xp + yq, Xf) - Xf\big) \equiv 2(ap + bq),$$
$$(xp, ap + bq) \equiv -ap, \ (yq, ap + bq) \equiv -bq,$$

we find that the group must contain one of the subgroups p, yq, or q, xp, which have just been considered.

Finally, let the subgroup $p + axq$, q belong to the group required. One of the additional infinitesimal transformations may be of the form

$$b(x^2p + xyq) + c(xyp + y^2q) + (gx + hy)p + (lx + ky)q \equiv Xf,$$

where b and c are not both zero.

Forming the commutators

$$\big(p + axq, (p + axq, Xf)\big) - (2b + ah)(p + axq)$$
$$\equiv -3a(b + ah)xq + 3acxp + aq \equiv Yf,$$
$$(p + axq, Yf) - 3ac(p + axq) \equiv -6a^2cxq + \beta q,$$
$$(q, Xf) - h(p + axq) \equiv (b - ah)xq + c(xp + 2yq) + lq,$$

we find that, if $a \neq 0$, we have the subgroup q, xq; and if $a = 0$, we see from

$$(-cp + bq, Xf) \equiv (bx + cy)(-cp + bq) + \gamma p + \delta q.$$

that we have the subgroup $-cp+bq$, $(bx+cy)(-cp+bq)$, which can be transformed into the subgroup q, xq by a linear transformation of the variables. But the primitive groups containing the subgroup q, xq have been determined above.

There remains to be determined the primitive groups whose infinitesimal transformations—at least four in number (§ 3)—are of the form

$$p+axq,\ q,\ (bx+cy)p+(ex+gy)q.$$

If we bear in mind that all the coefficients of p cannot be functions of x only, as we should then have an invariant differential equation of the first order $dx/dy=0$, and that all the coefficients of q cannot be functions of y only, as then the equation $dy/dx=0$ would be invariant, we find by building commutators that the first two groups of (12) will result. Accordingly:

All the primitive groups of the plane are represented by the types (12).*

* Cf. *Continuierliche Gruppen*, p. 360.